Math 1010C Term 1 2015
Supplementary exercises 5

1. Prove Leibniz's rule for higher order derivatives: if f, g are both n-times differentiable at a point a, then
$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a).$$

2. Suppose f is defined on an open interval I that contains a point a, and that f is differentiable on $I \setminus \{a\}$.
   (a) Show that each of the following statements is false:
      (i) If $\lim\limits_{x\to a} f'(x)$ exists, then f is differentiable at a.
      (ii) If $\lim\limits_{x\to a} f'(x)$ does not exist, then f is not differentiable at a.
   (b) Show, however, that the following statement is true:
   Suppose $\lim\limits_{x\to a} f'(x)$ exists. In addition, suppose f is continuous at a. Then f is differentiable at a, and $f'(a) = \lim\limits_{x\to a} f'(x)$.

3. The following gives a heuristic proof of the first form of L'Hopital's rule. The task here is to make precise the proof (for instance, by using the definition of limits).

   Suppose $f, g\colon (a, b) \to \mathbb{R}$ are differentiable on (a, b), with $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose also that
$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0,$$
   and that
$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} \quad \text{exists and equals } L.$$
   We will prove that
$$\lim_{x\to a^+} \frac{f(x)}{g(x)} \quad \text{also exists and equals } L.$$
   To do so, suppose $x \in (a, b)$. Let y be such that $a < y < x$. Then by Cauchy's mean value theorem, there exists $\xi \in (y, x)$ such that
$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}.$$
   Note ξ depends on both x and y. Now let $y \to a^+$. The left hand side converges to $f(x)/g(x)$, and the right hand side can be made arbitrarily close to L, as long as x is also sufficiently close to a. This suggests that $\lim_{x\to a^+} \frac{f(x)}{g(x)}$ also exists and equals L.

   Can you make precise the above argument?

4. The following gives a heuristic proof of the second form of L'Hopital's rule. The task here is to make precise the proof (for instance, by using the definition of limits).

   Suppose $f, g\colon (a,b) \to \mathbb{R}$ are differentiable on (a,b), with $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose also that

$$\lim_{x \to a^+} |g(x)| = +\infty,$$

   and that

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} \quad \text{exists and equals } L.$$

   We will prove that

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{also exists and equals } L.$$

   To do so, suppose $x \in (a,b)$. Let y be such that $x < y < b$. Then by Cauchy's mean value theorem, there exists $\xi \in (x,y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}.$$

   Note ξ depends on both x and y. Now let x be just slightly bigger than a. The left hand side is then approximately $f(x)/g(x)$, and the right hand side can be made arbitrarily close to L, as long as y is also sufficiently close to a. This suggests that $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ also exists and equals L.

   Can you make precise the above argument?

L' Hospital's Rule (Second form $\frac{\infty}{\infty}$). (Tutorial 7)

Suppose $f, g : (a,b) \longrightarrow \mathbb{R}$ is differentiable st. $\lim_{x\to a^+} f(x) = \infty = \lim_{x\to a^+} g(x)$.

If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ exists $=: L$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.

Proof: We need to borrow some $y \in (a,b)$.

($\bigstar$) Note that for each $y \in (a,b)$, $\lim_{x\to a^+} \frac{f(x)-f(y)}{g(x)-g(y)} = \lim_{x\to a^+} \frac{f(x)}{g(x)}$ if any one of them exists, but that is not enough as we shall see in our proof.

We borrow $y \in (a,b)$ and let $x \to a^+$:

Fix $y \in (a,b)$, $(x<y)$, $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(\xi_x)}{g'(\xi_x)}$ for some $\xi_x \in (x,y)$. $\left(\begin{matrix}\text{Cauchy} \\ \text{MVT}\end{matrix}\right)$.

Since $\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ exists $=: L$; $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(\xi_x)}{g'(\xi_x)}$ is close to L whenever you have fixed y which is close to a, no matter what x you take (however $x<y$ is a must)

The proof finishes if one can show that (Compare with $\bigstar$).

For each fixed $y \in (a,b)$, $\mathcal{E}_y(x) := \frac{f(x)}{g(x)} - \frac{f(x)-f(y)}{g(x)-g(y)} \longrightarrow 0$ as $x \to a^+$

$$\mathcal{E}_y(x) = \frac{f(x)}{g(x)} - \frac{f(x)-f(y)}{g(x)-g(y)} = \frac{-f(x)g(y) + f(y)g(x)}{g(x)(g(x)-g(y))} = \frac{-f(x)\,g(y)}{g(x)(g(x)-g(y))} + \boxed{\frac{f(y)}{g(x)-g(y)}}^{\text{OK}}$$

$$= \boxed{\frac{-g(y)(f(x)-f(y)) - \boxed{f(y)g(y)}^{\text{OK}}}{g(x)(g(x)-g(y))}} + \boxed{\text{OK}}$$

$$= -\frac{g(y)}{g(x)}\left(\frac{f(x)-f(y)}{g(x)-g(y)}\right) + \boxed{\text{OK}}$$

But $\frac{f(x)-f(y)}{g(x)-g(y)}$ is close to L, whenever y is cleverly chosen,

in particular $\left|\frac{f(x)-f(y)}{g(x)-g(y)}\right| \le M \quad \forall\, x<y$ and $\frac{g(y)}{g(x)} \to 0$ as $x \to a^+$

Everything is OK.

Why it is done ?? $\frac{f(x)}{g(x)} = \frac{f(x)-f(y)}{g(x)-g(y)} + \mathcal{E}_y(x) = \frac{f'(\xi_x)}{g'(\xi_x)} + \mathcal{E}_y(x)$ for fixed $y \in (a,b)$

$\frac{f'(\xi_x)}{g'(\xi_x)}$ is close to L when y is cleverly chosen. $\mathcal{E}_y(x)$ is close to 0 as x is close to a.

So, $\frac{f(x)}{g(x)}$ is close to L as x is close to a.

One can also show that ① $\boxed{\frac{f(x)}{g(x)} \to \infty \text{ as } x \to a^+ \text{ if } \frac{f'(z)}{g'(z)} \to \infty \text{ as } z \to a^+}$

But now we cannot argue $\frac{f(x)}{g(x)} - \frac{f(x)-f(y)}{g(x)-g(y)} \to 0$ as $x \to a^+$ as before.

Try to argue ①, it is easy.

Proof of L' Hospital Rule is about the art of borrowing:

$f(x), g(x) \to 0$ as $x \to a^+$, we borrow $y = a$.

$f(x), g(x) \to 0$ as $x \to \infty$, we borrow $y = \infty$??

$f(x), g(x) \to \infty$ as $x \to a^+$, ??

$f(x), g(x) \to \infty$ as $x \to \infty$, ??

Math 1010C Term 1 2015
Supplementary exercises 6

1. The following theorem is often useful in computing Taylor polynomials of products / quotients / compositions of functions.

   Theorem 1. *Let n be a non-negative integer. Suppose f is a function defined on an open interval I containing a point c, and that f is n-times differentiable on I. Assume that there exists a polynomial P_n of degree $\leq n$, and a function E_n defined on I, such that*

   $$f(x) = P_n(x) + E_n(x) \quad \text{for all } x \in I, \text{ with} \quad \lim_{x \to c} \frac{E_n(x)}{x^n} = 0.$$

   Then $P_n(x)$ is the degree n Taylor polynomial of f centered at c.

   The goal of this question is to establish this result.

   (a) Let k be a positive integer. Suppose f is a function defined on an open interval I containing 0, and that f is k-times differentiable on I. Show that

   $$\lim_{x \to 0} \frac{f(x) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} x^j}{x^k} \quad \text{exists, and is equal to} \quad \frac{f^{(k)}(0)}{k!}.$$

   (Hint: Apply L'Hopital's rule $(k-1)$ times, and then use the definition of $f^{(k)}(0)$.)

   (b) Let n be a non-negative integer. Suppose f is a function defined on an open interval I containing 0, and that f is n-times differentiable on I. Assume that there exists a polynomial P_n of degree $\leq n$, and a function E_n defined on I, such that

   $$f(x) = P_n(x) + E_n(x) \quad \text{for all } x \in I, \text{ with} \quad \lim_{x \to 0} \frac{E_n(x)}{x^n} = 0.$$

   Show that

   (i) $\displaystyle\lim_{x \to 0} \frac{E_n(x)}{x^k} = 0$ for any non-negative integer $k \leq n$.

   (ii) $f^{(k)}(0) = P_n^{(k)}(0)$ for any non-negative integer $k \leq n$. (Hint: We proceed by induction on k. For $k = 0$, just recall $f(x) = P_n(x) + E_n(x)$, and let $x \to 0$. Assume now for some positive integer $k \leq n$, we have

   $$\begin{cases} f(0) = P_n(0), \\ f'(0) = P_n'(0), \\ \qquad \vdots \\ f^{(k-1)}(0) = P_n^{(k-1)}(0). \end{cases}$$

   We want to prove that $f^{(k)}(0) = P_n^{(k)}(0)$. But then by induction hypothesis,

   $$\frac{f(x) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} x^j}{x^k} = \frac{P_n(x) - \sum_{j=0}^{k-1} \frac{P_n^{(j)}(0)}{j!} x^j}{x^k} + \frac{E_n(x)}{x^k}$$

   for all $x \in I \setminus \{0\}$. Since both f and P_n are k-times differentiable, letting $x \to 0$ and using (a), we get our desired conclusion.)

   (iii) P_n is the degree n Taylor polynomial of f centered at 0. (Hint: It suffices to show that for any polynomial P of degree n, we have

   $$\sum_{k=0}^{n} \frac{P^{(k)}(0)}{k!} x^k = P(x).$$

But if $P(x) = \sum_{k=0}^{n} a_k x^k$ for some coefficients $a_0, a_1, \dots, a_n$, then differentiating both sides k times and setting $x = 0$, we get

$$a_k = \frac{P^{(k)}(0)}{k!}$$

for any non-negative integer $k \leq n$. This concludes the proof.)

2. Below we see some applications of the earlier question to the computation of some Taylor polynomials.
   (a) The goal in this part is to compute the degree 23 Taylor polynomial of $\cosh(x^3)$ centered at 0.
      (i) Show that there exists a function A, defined on $\mathbb{R}$, such that

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + A(x) \quad \text{for all } x \in \mathbb{R}, \text{ with} \quad \lim_{x \to 0} \frac{A(x)}{x^\alpha} = 0 \quad \text{for any } \alpha < 8.$$

      (ii) Show that there exists a function B, defined on $\mathbb{R}$, such that

$$\cosh(x^3) = 1 + \frac{x^6}{2!} + \frac{x^{12}}{4!} + \frac{x^{18}}{6!} + B(x) \quad \text{for all } x \in \mathbb{R}, \text{ with} \quad \lim_{x \to 0} \frac{B(x)}{x^{23}} = 0.$$

      Hence find the degree 23 Taylor polynomial of $\cosh(x^3)$ centered at 0.

   (b) The goal in this part is to compute the degree 4 Taylor polynomial of $e^{-2x} \sin x$ centered at 0.
      (i) Show that there exists a function A, defined on $\mathbb{R}$, such that

$$e^{-2x} = 1 - 2x + 2x^2 - \frac{4x^3}{3} + A(x) \quad \text{for all } x \in \mathbb{R}, \text{ with} \quad \lim_{x \to 0} \frac{A(x)}{x^3} = 0.$$

      (ii) Show that there exists a function B, defined on $\mathbb{R}$, such that

$$\sin x = x - \frac{x^3}{6} + B(x) \quad \text{for all } x \in \mathbb{R}, \text{ with} \quad \lim_{x \to 0} \frac{B(x)}{x^4} = 0.$$

      (iii) Show that there exists a function C, defined on $\mathbb{R}$, such that

$$e^{-2x} \sin x = \left(1 - 2x + 2x^2 - \frac{4x^3}{3}\right)\left(x - \frac{x^3}{6}\right) + C(x) \quad \text{for all } x \in \mathbb{R}, \text{ with} \quad \lim_{x \to 0} \frac{C(x)}{x^4} = 0.$$

      Hence find the degree 4 Taylor polynomial of $e^{-2x} \sin x$ centered at 0.

   (c) The goal in this part is to compute the degree 5 Taylor polynomial of $\sec x = \dfrac{1}{\cos x}$ centered at 0.
      (i) Show that there exists a function A, defined on $\mathbb{R}$, such that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + A(x) \quad \text{for all } x \in \mathbb{R}, \text{ with} \quad \lim_{x \to 0} \frac{A(x)}{x^5} = 0.$$

      (ii) Show that there exists a function B, defined on $\mathbb{R}$, such that

$$\frac{1}{\cos x} = 1 + \left(\frac{x^2}{2!} - \frac{x^4}{4!}\right) + \left(\frac{x^2}{2!} - \frac{x^4}{4!}\right)^2 + B(x) \quad \text{for all } x \in \mathbb{R}, \text{ with} \quad \lim_{x \to 0} \frac{B(x)}{x^5} = 0.$$

      Hence find the degree 5 Taylor polynomial of $\sec x$ centered at 0.

   (d) Can you now combine the techniques in parts (b) and (c), to compute the degree 5 Taylor polynomial of $\tan x = \dfrac{\sin x}{\cos x}$ centered at 0?

Remark. The above may not be the fastest way of computing the Taylor polynomial of $\sec x$ or $\tan x$ centered at 0. One may want to take instead the identities $\sec x \cos x = 1$ and $\tan x \cos x = \sin x$, differentiate them using Leibniz's rule, and evaluate at 0 to compute the higher order derivatives of $\sec x$ and $\tan x$ at 0, thereby yielding the Taylor polynomial of $\sec x$ and $\tan x$ centered at 0.